Defect Subspaces and Localized Instabilities in Cut-Cell Finite-Volume Operators

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Abstract

Cut-cell instability in explicit finite-volume methods is a low-dimensional geometric phenomenon. For a periodic upwind operator with m small cut cells, the unstable spectral subspace $\mathcal{U}$ is m-dimensional, one mode per defect, and converges to the coordinate subspace $\mathcal{C} = \text{span}\{e_{j_1}, \ldots, e_{j_m}\}$ as the cut-cell volume fractions tend to zero. We prove this for the one-cut-cell one-dimensional case via an explicit characteristic polynomial, and for the one-cut-cell d-dimensional case via a block-Schur argument. The unstable eigenvector satisfies $v_u = e_c + O(\alpha)$, localizing to the cut cell with exponential decay in distance. For merge-left state redistribution, the correction direction satisfies $|\cos_\alpha(D, v_u)| = 1 - O(\alpha)$: SRD acts asymptotically in the unstable eigendirection. This identifies the stabilization mechanism as spectral targeting of the defect subspace. The singular-block operator framework extends these results to a broad class of explicit finite-volume schemes. We prove that the defect matrix is given by $\Gamma = (\Delta t/h)(a_{c+1/2,c} - a_{c-1/2,c})$, the net linearized diagonal flux coefficient at the cut-cell faces scaled by $\Delta t/h$, computable from local geometry alone. This is verified for first-order upwind, scalar Roe/Godunov, and MUSCL with bounded reconstruction stencils. As a consequence, the minimum stabilizing blending parameter for merge-left SRD is $s_{\text{stab}} = (\lambda_\alpha - 2)/(\lambda_\alpha - 1) + O(\alpha)$, with $\lambda_\alpha = \Gamma/\alpha$, requiring no global eigenvalue computation. For multiple separated cut cells, the one-cut-cell theorem applies locally at each defect; separation and localization imply one unstable mode per cut cell.

1 Introduction

Small cut cells are a persistent difficulty in explicit finite-volume methods for embedded boundary problems. A cut cell of volume fraction $\alpha \ll 1$ forces the local CFL number $\lambda_\alpha = \lambda/\alpha$ to be large even when the background CFL λ is stable, and the resulting instability can destroy the computation in a single timestep. Stabilization methods including state redistribution [Berger and Giuliani, 2021], flux redistribution, and cell merging have been developed to address this, but the spectral mechanism underlying both the instability and its correction has not been characterized.

This paper establishes that mechanism. The central claim is that cut-cell instability is generated by a low-dimensional defect subspace: for m small cut cells, the update operator has exactly m large unstable eigenvalues, each localized at one cut cell. The unstable spectral subspace $\mathcal{U}$ converges to the coordinate subspace $\mathcal{C}$ spanned by the defect cells as the volume fractions tend to zero. Stabilization methods that act locally at cut cells, such as state redistribution, damp the unstable modes directly because their corrections are asymptotically aligned with the defect subspace.

The argument proceeds through a chain:

$$\text{small-cell geometry} \longrightarrow \Gamma \longrightarrow \mathcal{U} \longrightarrow \text{stabilization}. \tag{1}$$

The cut-cell geometry determines a defect matrix Γ. The defect matrix determines the unstable spectrum. The unstable spectrum determines where stabilization must act.

The paper is organized as follows, with an explicit statement of rigor level for each part. Sections 2–3 are fully rigorous: Section 2 gives the complete one-cut-cell 1D analysis via an explicit characteristic polynomial, and Section 3 proves the SRD alignment theorem. Section 4 is essentially rigorous, with the interaction between cut cells handled via Lemma 4.1 (Kato's persistence theorem) under a quantitative separation assumption. Section 5 proves the one-cut-cell d-dimensional result; the stability equivalence holds under the additional assumption that the background scheme is stable. Section 6 proves the abstract singular-block spectral theorem fully. Section 7 proves that $\Gamma = (\Delta t/h)(a_{c+1/2,c} - a_{c-1/2,c})$ is computable directly from local flux geometry (Theorem 7.1), and verifies the framework for first-order upwind, scalar Roe/Godunov, and MUSCL. Section 8 derives the geometry-based stability criterion for the one-cut-cell 1D blended scheme. Section 9 concludes.

2 One-Cut-Cell One-Dimensional Analysis

2.1 Setup

Consider the periodic upwind operator on N cells with one cut cell at position c of volume fraction $\alpha \in (0,1)$:

$$(Au)_i = (1-\lambda_i)u_i + \lambda_i u_{i-1}, \qquad \lambda_i = \begin{cases} \lambda_\alpha = \lambda/\alpha & i = c, \\ \lambda & i \neq c, \end{cases} \tag{2}$$

with periodic indexing. Write $A = B + (\lambda_\alpha - \lambda)e_c(e_{c-1} - e_c)^\top$ where B is the uniform upwind circulant at CFL λ.

2.2 Characteristic Polynomial

The eigenvalue equation $Av = \mu v$ at a non-cut cell $i \neq c$ gives the recurrence

$$v_i = q\, v_{i-1}, \qquad q = \frac{\lambda}{\mu - 1 + \lambda}. \tag{3}$$

Hence $v_i = q^i v_0$. Substituting into the cut-cell row and using periodicity $v_{N-1} = q^{N-1}v_0$ yields, after eliminating μ via $\mu = 1 - \lambda + \lambda/q$, the characteristic equation

$$P_\alpha(q) = q^N - (1-\alpha)q - \alpha = 0. \tag{4}$$

All eigenvalues of A correspond to roots of P_α via $\mu(q) = 1 - \lambda + \lambda/q$.

2.3 Distinguished Small Root

The linear part of P_α vanishes at

$$q_0 = -\frac{\alpha}{1-\alpha}. \tag{5}$$

Setting $q = q_0 + \eta$, the equation $P_\alpha(q) = 0$ becomes

$$\eta = \frac{(q_0 + \eta)^N}{1-\alpha}. \tag{6}$$

The map $T(\eta) = (q_0 + \eta)^N/(1-\alpha)$ satisfies $|T'(\eta)| = N|q_0 + \eta|^{N-1}/(1-\alpha) = O(\alpha^{N-1})$ on any fixed ball, so by the contraction mapping theorem there is a unique root

$$q_u = q_0 + O(\alpha^N) = -\frac{\alpha}{1-\alpha} + O(\alpha^N). \tag{7}$$

2.4 Dominant Eigenvalue

From $\mu_u = 1 - \lambda + \lambda/q_u$ and $1/q_u = 1/q_0 + O(\alpha^{N-1}) = -(1-\alpha)/\alpha + O(\alpha^{N-1})$:

$$\mu_u = 1 - \lambda - \frac{\lambda(1-\alpha)}{\alpha} + O(\alpha^{N-2}) = 1 - \frac{\lambda}{\alpha} + O(\alpha^{N-2}) = 1 - \lambda_\alpha + O(\alpha^{N-2}). \tag{8}$$

2.5 Dominance

By Rouché's theorem applied to $P_\alpha(q) = q^N - (1-\alpha)q - \alpha$ on the circle $|q| = 1/2$: for small α, $|q^N| < |(1-\alpha)q + \alpha|$ on $|q| = 1/2$, so P_α has exactly one root inside $|q| = 1/2$, which is q_u. The remaining $N-1$ roots satisfy $|q| \geq 1/2$ and their eigenvalues $\mu(q) = 1 - \lambda + \lambda/q$ are $O(1)$. Since $|\mu_u| = \lambda_\alpha - 1 + O(\alpha^{N-2}) \to \infty$ as $\alpha \to 0$, μ_u is the dominant eigenvalue.

2.6 Instability Threshold

Corollary 2.1 (Instability threshold)**.** *The base scheme is spectrally stable, $\rho(A) \leq 1$, if and only if $\lambda_\alpha \leq 2$. For $\lambda_\alpha > 2$, the dominant eigenvalue $\mu_u = 1 - \lambda_\alpha + O(\alpha^{N-2})$ lies outside the unit disk.*

Proof. From (8), $|\mu_u| = \lambda_\alpha - 1 + O(\alpha^{N-2})$ for $\lambda_\alpha > 1$. Stability requires $|\mu_u| \leq 1$, i.e., $\lambda_\alpha - 1 \leq 1$, i.e., $\lambda_\alpha \leq 2$. □

2.7 Eigenvector Localization

Theorem 2.2 (Eigenvector localization, 1D)**.** *Normalize the unstable eigenvector by $(v_u)_c = 1$. Then*

$$v_u = e_c + O(\alpha) \tag{9}$$

in the Euclidean norm.

Proof. From the recurrence (3) with eigenvalue μ_u, the ratio $q = \lambda/(\mu_u - 1 + \lambda)$. Substituting $\mu_u = 1 - \lambda_\alpha + O(\alpha^{N-2})$ gives $q = -\alpha/(1-\alpha) + O(\alpha^{N-1}) = O(\alpha)$. Entry i of v_u satisfies $|(v_u)_i| = |q|^{|i-c|} = O(\alpha^{|i-c|})$. Therefore

$$|v_u - e_c|_2^2 = \sum_{i \neq c} |(v_u)_i|^2 = O(\alpha^2),$$

giving $|v_u - e_c|_2 = O(\alpha)$. □

The localization is exponential in distance from c: entry i decays as $O(\alpha^{|i-c|})$. For $\alpha = 0.1$ the entry two cells from the cut cell is $O(0.01)$.

3 SRD Alignment Theorem

Merge-left state redistribution on the neighborhood $\{c-1, c\}$ produces a correction vector

$$D = \Delta(\alpha e_{c-1} - e_c), \qquad \Delta = \frac{\lambda p + (1-\lambda_\alpha) r}{1+\alpha}, \tag{10}$$

where $p = U^n_{c-1} - U^n_{c-2}$ and $r = U^n_c - U^n_{c-1}$. This follows directly from the merge-left SRD construction; see Karell [2025b].

Use the volume-weighted inner product

$$\langle x, y\rangle_\alpha = \alpha x_c y_c + \sum_{i \neq c} x_i y_i. \tag{11}$$

Theorem 3.1 (SRD alignment)**.** *For $\Delta \neq 0$,*

$$|\cos_\alpha(D, v_u)| = 1 - O(\alpha). \tag{12}$$

Proof. From (10), $D_c = -\Delta$ and $D_{c-1} = \alpha\Delta$, so D is supported on $\{c-1, c\}$. With $(v_u)_c = 1$ and $(v_u)_{c-1} = q_u = -\alpha/(1-\alpha) + O(\alpha^N) = O(\alpha)$:

$$\langle D, v_u\rangle_\alpha = \underbrace{w_{c-1}}_{=1} D_{c-1}(v_u)_{c-1} + \underbrace{w_c}_{=\alpha} D_c(v_u)_c = 1 \cdot (\alpha\Delta) \cdot O(\alpha) + \alpha \cdot (-\Delta) \cdot 1 = -\alpha\Delta + O(\alpha^2\Delta), \tag{13}$$

$$|D|^2_\alpha = (\alpha\Delta)^2 + \alpha\Delta^2 = \alpha(1+\alpha)\Delta^2, \tag{14}$$

$$|v_u|^2_\alpha = \alpha \cdot 1^2 + 1 \cdot O(\alpha)^2 = \alpha + O(\alpha^2). \tag{15}$$

Therefore

$$\cos_\alpha(D, v_u) = \frac{-\alpha\Delta + O(\alpha^2\Delta)}{|\Delta|\sqrt{\alpha(1+\alpha)} \cdot \sqrt{\alpha + O(\alpha^2)}} = \frac{-1 + O(\alpha)}{\sqrt{(1+\alpha)(1+O(\alpha))}} = -1 + O(\alpha).$$

Taking absolute value gives (12). □

The theorem says SRD acts asymptotically in the unstable eigendirection. For $\alpha = 0.1$ the cosine is within $O(0.1)$ of ± 1; for $\alpha = 0.01$ within $O(0.01)$. The $O(\alpha)$ gap comes from the nonzero $c-1$ entry of v_u, which is $O(\alpha)$ rather than zero.

4 Multiple Cut Cells

Let $J = \{j_1, \dots, j_m\}$ be the set of cut cells, with volume fractions $\alpha_1, \dots, \alpha_m$ and local CFL numbers $\lambda_k = \lambda/\alpha_k$. Define the defect coordinate subspace

$$\mathcal{C} = \operatorname{span}\{e_{j_1}, \dots, e_{j_m}\}. \tag{16}$$

The proof of the multi-cut-cell theorem uses the following standard perturbation result.

Lemma 4.1 (Persistence of simple eigenvalues, [Kato, 1966, Thm. II.5.4])**.** *Let A_0 have a simple eigenvalue μ_0, and let $A = A_0 + E$ with $\|E\| \le \delta$. If $\delta < \operatorname{dist}(\mu_0, \sigma(A_0) \setminus \{\mu_0\})/4$, then A has exactly one eigenvalue μ with $|\mu - \mu_0| \le C\delta$, and the associated spectral projector satisfies $\|P_\mu - P_{\mu_0}\| \le C\delta$.*

Theorem 4.2 (Separated multi-cut-cell instability)**.** *Assume the cut cells are non-adjacent and that the local cut-cell CFLs are separated:*

$$|\lambda_k - \lambda_\ell| \geq c_0 > 0, \qquad k \neq \ell. \tag{17}$$

Then the one-cut-cell analysis of Section 2 applies locally at each cut cell j_k, giving asymptotically

$$\mu_k = 1 - \lambda_k + O(1), \qquad k = 1, \ldots, m, \tag{18}$$

with eigenvectors satisfying $v_k = e_{j_k} + O(\alpha_k)$ locally near j_k. Under the exponential localization of Theorem 2.2 and the separation condition, the interaction between distinct defect modes is exponentially small in α_k, and the m localized eigenpairs persist with corrections of order $O(\alpha_k^{d_{\min}})$ where $d_{\min} \geq 2$ is the minimum inter-cell separation. Consequently, the unstable spectral subspace satisfies

$$\mathcal{U} \to \mathcal{C} \qquad \text{as } \max_k \alpha_k \to 0. \tag{19}$$

Remark on rigor. *The one-cut-cell results and the interaction estimate (Lemma 4.3) are fully proved. The conclusion that each unstable eigenpair of $A^{(k)}$ persists in the full operator requires controlling the eigenvalue movement under the perturbation $E^{(k)}$. For non-normal operators a small residual $\|E^{(k)}v_k\|$ does not automatically imply a nearby eigenvalue without controlling the eigenpair condition number. This step is therefore stated as an asymptotic result supported by the localization and interaction estimates, rather than a fully rigorous perturbation theorem.*

Proof. At cut cell j_k, the first-order upwind row is

$$(A_{\boldsymbol{\alpha}} u)_{j_k} = (1 - \lambda_k) u_{j_k} + \lambda_k u_{j_k - 1}, \qquad \lambda_k = \frac{\lambda}{\alpha_k}.$$

The diagonal defect at each cut cell has size λ/α_k, so the actual defect structure is

$$\operatorname{diag}\left(\frac{\lambda}{\alpha_1}, \ldots, \frac{\lambda}{\alpha_m}\right) + O(1).$$

This is not a single singular-block decomposition with one small parameter. In the one-cut-cell convention, the k-th local block has singular form

$$A_{\alpha_k}^{(k)} = B^{(k)} - \frac{1}{\alpha_k}\Gamma_k + O(1), \qquad \Gamma_k = \lambda.$$

We therefore apply the one-cut-cell theorem (Corollary 2.1 and Theorem 2.2) separately at each cut cell j_k with $\alpha = \alpha_k$, giving

$$\mu_k = 1 - \frac{\lambda}{\alpha_k} + O(1), \qquad v_k = e_{j_k} + O(\alpha_k).$$

It remains to bound the effect of the other cut cells on the j_k-localized eigenpair. The operator A decomposes as $A = A^{(k)} + E^{(k)}$, where $A^{(k)}$ is the isolated one-cut-cell operator at j_k (with all other cut cells replaced by background cells) and $E^{(k)}$ collects the rank-one corrections at all other cut cells j_ℓ, $\ell \neq k$.

Lemma 4.3 (Localized mode interaction estimate)**.** *For non-adjacent cut cells with minimum separation $d_{\min} = \min_{k \neq \ell} |j_k - j_\ell| \geq 2$, and for the j_k-localized unstable eigenvector v_k of $A^{(k)}$ normalized by $(v_k)_{j_k} = 1$,*

$$\|E^{(k)} v_k\|_2 = O(\alpha_k^{d_{\min}}),$$

uniformly in N.

Proof. Each cut cell j_ℓ, $\ell \neq k$, contributes a rank-one correction $R_\ell = (\lambda_\ell - \lambda)(e_{j_\ell} e_{j_\ell - 1}^\top - e_{j_\ell} e_{j_\ell}^\top)$ to $E^{(k)}$. Note $\|R_\ell\|_2 = O(\lambda_\ell) = O(\alpha_\ell^{-1})$ is large; however, the action on v_k depends only on the entries of v_k at j_ℓ and $j_\ell - 1$. These cells are at distance at least $d_{\min}$ from j_k, so by Theorem 2.2,

$$|(v_k)_{j_\ell}|,\ |(v_k)_{j_\ell - 1}| = O(\alpha_k^{d_{\min}}).$$

Therefore $\|R_\ell v_k\|_2 = O(\lambda_\ell \alpha_k^{d_{\min}})$, and summing over $\ell \neq k$ gives $\|E^{(k)} v_k\|_2 = O((m-1)\lambda_{\max}\alpha_k^{d_{\min}}) = O(\alpha_k^{d_{\min}})$. The bound is uniform in N: periodic wrap contributes $O(\alpha_k^{N-d_{\min}})$, which is negligible. □

Lemma 4.3 shows the interaction between distinct defect modes is exponentially small: the perturbation $E^{(k)}$ acts on the localized eigenvector v_k with magnitude $O(\alpha_k^{d_{\min}})$. Combined with the separation condition (17), which keeps the unstable eigenvalues mutually separated by at least c_0, this supports the conclusion that the m localized eigenpairs persist with corrections of order $O(\alpha_k^{d_{\min}})$. As noted in the theorem statement, making this final step fully rigorous for a non-normal operator requires controlling the eigenpair condition number, which is not done here.

The corresponding eigenvectors have disjoint leading supports: $v_k = e_{j_k} + O(\alpha_k)$. Therefore

$$\operatorname{span}\{v_1, \dots, v_m\} \to \operatorname{span}\{e_{j_1}, \dots, e_{j_m}\} = \mathcal{C},$$

which proves $\mathcal{U} \to \mathcal{C}$. □

Remark 4.4. *The defect matrix notation in the multi-cut-cell case must be used with care. The actual diagonal defect is* $\operatorname{diag}(\lambda/\alpha_1, \dots, \lambda/\alpha_m) + O(1)$*, whereas the one-cut-cell singular-block theorem uses an* $O(1)$ *defect coefficient* $\Gamma_k = \lambda$ *after factoring out* $1/\alpha_k$*. Theorem 6.1 should not be invoked with a single global* α*. The correct argument is local: apply the one-cut-cell result to each cut cell and use separation and localization to assemble the* m *modes.*

5 One-Cut-Cell d-Dimensional Case

Consider a d-dimensional periodic upwind discretization on a Cartesian mesh with CFL numbers $\lambda_1, \dots, \lambda_d$ in each coordinate direction. Let $L = \sum_{r=1}^d \lambda_r$. For a single cut cell of volume fraction α, the cut-cell row of the update operator is

$$(Au)_c = \left(1 - \frac{L}{\alpha}\right) u_c + \sum_{r=1}^{d} \frac{\lambda_r}{\alpha} u_{c-\hat{r}}, \tag{20}$$

where $c - \hat{r}$ denotes the upwind neighbor in coordinate direction r.

Theorem 5.1 (One-cut-cell d-dimensional)**.** *The operator possesses a unique large eigenvalue*

$$\mu_\alpha = 1 - \frac{L}{\alpha} + O(1), \tag{21}$$

with eigenvector $v_\alpha = e_c + O(\alpha)$*. This large eigenvalue satisfies* $|\mu_\alpha| > 1$ *if and only if* $L/\alpha > 2$*. All other eigenvalues remain* $O(1)$*; when* $\lambda_r \leq 1$ *for all* r *the background circulant eigenvalues satisfy* $|\mu_k| \leq 1$*, so the operator is spectrally unstable if and only if* $L/\alpha > 2$*.*

Proof. The cut-cell row (20) has the block form

$$A = \begin{pmatrix} 1 - L/\alpha & L/\alpha\, b^\top \\ 0 & B \end{pmatrix} + O(1), \tag{22}$$

where $b^\top = (\lambda_1/L, \ldots, \lambda_d/L, 0, \ldots)$ selects the upwind neighbors and B is the background circulant block with eigenvalues $O(1)$. The large eigenvalue comes from the Schur complement of the (c, c) entry. For the cut-cell entry, $\mu_\alpha = 1 - L/\alpha + b^\top(\mu_\alpha I - B)^{-1}(L/\alpha)b$. Since B has eigenvalues $O(1)$ and $\mu_\alpha = O(1/\alpha)$, $\|(\mu_\alpha I - B)^{-1}\| = O(|\mu_\alpha|^{-1}) = O(\alpha)$, so $b^\top(\mu_\alpha I - B)^{-1}(L/\alpha)b = O(\alpha)\cdot O(\alpha^{-1}) = O(1)$, giving (21). The eigenvector equation at non-cut cells is $(\mu_\alpha - 1 + \lambda_r)v_{c-\hat{r}} = \lambda_r v_c$, giving $v_{c-\hat{r}} = \lambda_r/(\mu_\alpha - 1 + \lambda_r) \cdot v_c = O(\alpha)$ since $\mu_\alpha - 1 = O(1/\alpha)$. Hence $v_\alpha = e_c + O(\alpha)$ in Euclidean norm. □

The stability threshold $\alpha \geq L/2$ reduces to $\lambda_\alpha \leq 2$ in 1D (where $L = \lambda$) and generalizes it: in d dimensions, the cut cell is stable when its volume fraction exceeds half the sum of the directional CFL numbers.

6 Singular-Block Framework

The one-cut-cell results suggest a general framework. A raw cut-cell finite-volume update operator on a mesh with small cells has the form, after reordering unknowns into defect coordinates (cut cells) and regular coordinates:

$$A_\alpha = \begin{pmatrix} -\alpha^{-1}\Gamma + R_\alpha & B_\alpha \\ P_\alpha & Q_\alpha \end{pmatrix}, \tag{23}$$

where Γ is the defect matrix encoding cut-cell geometry, $|R_\alpha| = O(1)$, $|Q_\alpha| = O(1)$, $|P_\alpha| = O(1)$, and $|B_\alpha| = O(\alpha^{-1})$.

Theorem 6.1 (Singular-block spectral theorem)**.** *Let $\gamma_1, \ldots, \gamma_m$ be the simple nonzero eigenvalues of Γ. Then A_α possesses exactly m eigenvalues satisfying*

$$\mu_k(\alpha) = -\frac{\gamma_k}{\alpha} + O(1), \qquad k = 1, \ldots, m, \tag{24}$$

and all remaining eigenvalues are $O(1)$.

Proof. Let μ be an eigenvalue with $|\mu| > |Q_\alpha| + 1$, so $\mu I - Q_\alpha$ is invertible. The Schur complement factorization gives

$$\det(\mu I - A_\alpha) = \det(\mu I - Q_\alpha)\det S_\alpha(\mu),$$

where

$$S_\alpha(\mu) = \mu I_m + \alpha^{-1}\Gamma - R_\alpha - B_\alpha(\mu I - Q_\alpha)^{-1}P_\alpha.$$

Every large eigenvalue is a zero of $\det S_\alpha(\mu) = 0$.

Set $\nu = \alpha\mu$. For ν in a compact set $K \subset \mathbb{C} \setminus \{0\}$, $\mu = \nu/\alpha$ and

$$\left| \left(\tfrac{\nu}{\alpha} I - Q_\alpha\right)^{-1} \right| \leq C_K \alpha$$

uniformly, since $|Q_\alpha| \leq C$. Therefore

$$\left| B_\alpha \left(\tfrac{\nu}{\alpha} I - Q_\alpha\right)^{-1} P_\alpha \right| \leq C\alpha^{-1} \cdot C_K\alpha \cdot C = O(1).$$

Multiplying $S_\alpha(\nu/\alpha)$ by α and defining

$$E_\alpha(\nu) = -R_\alpha - B_\alpha\big(\tfrac{\nu}{\alpha}I - Q_\alpha\big)^{-1} P_\alpha,$$

we obtain $\alpha S_\alpha(\nu/\alpha) = \nu I_m + \Gamma + \alpha E_\alpha(\nu)$ with $|E_\alpha(\nu)| = O(1)$ uniformly on compact subsets of $\mathbb{C} \setminus \{0\}$. Define

$$F_\alpha(\nu) = \det\big(\nu I_m + \Gamma + \alpha E_\alpha(\nu)\big), \qquad F_0(\nu) = \det(\nu I_m + \Gamma).$$

Then $F_\alpha \to F_0$ uniformly on compact subsets of $\mathbb{C} \setminus \{0\}$. The zeros of F_0 are $\nu_k^0 = -\gamma_k$, simple by assumption. Fix k and choose a circle $|\nu + \gamma_k| = r$ containing no other zero of F_0 and not enclosing 0. For small α, $|F_\alpha - F_0| < |F_0|$ on this circle, so by Rouché's theorem F_α has exactly one zero near $-\gamma_k$.

To locate it precisely: since $-\gamma_k$ is a simple zero, $F_0(\nu) = F_0'(-\gamma_k)(\nu + \gamma_k) + O(|\nu + \gamma_k|^2)$ with $F_0'(-\gamma_k) \neq 0$, and $|F_\alpha(\nu) - F_0(\nu)| = O(\alpha)$ uniformly near $-\gamma_k$. On the circle $|\nu + \gamma_k| = M\alpha$, $|F_0(\nu)| \geq cM\alpha$ while $|F_\alpha - F_0| \leq C\alpha$. Choosing $M > C/c$, Rouché gives one zero inside, so $\nu_k(\alpha) = -\gamma_k + O(\alpha)$ and $\mu_k(\alpha) = -\gamma_k/\alpha + O(1)$.

No other unbounded eigenvalues. Since $|A_\alpha| = O(\alpha^{-1})$, every eigenvalue satisfies $|\alpha\mu| \leq C$. Let μ_α be any unbounded eigenvalue with $\nu_\alpha = \alpha\mu_\alpha$. If $\nu_\alpha \to \nu_0 \neq 0$, the Schur equation gives $F_0(\nu_0) = 0$, so $\nu_0 = -\gamma_k$ for some k, already counted. It remains to exclude $\nu_\alpha \to 0$ with $|\mu_\alpha| \to \infty$. In this case $|\mu_\alpha| \to \infty$ with $|\mu_\alpha| = o(\alpha^{-1})$, so $|(\mu_\alpha I - Q_\alpha)^{-1}| = O(|\mu_\alpha|^{-1})$ and

$$S_\alpha(\mu_\alpha) = \alpha^{-1}\Gamma + o(\alpha^{-1}).$$

Since Γ is invertible, $S_\alpha(\mu_\alpha)$ is invertible for small α, contradicting $\det S_\alpha(\mu_\alpha) = 0$. Hence no such eigenvalues exist, and A_α has exactly m unbounded eigenvalues. □

Theorem 6.2 (Defect subspace convergence)**.** *Under Theorem 6.1, the unstable spectral projector satisfies*

$$|P_{\mathcal{U}_\alpha} - P_{\mathcal{C}}| = O(\alpha), \tag{25}$$

where $\mathcal{C} = \mathbb{C}^m \oplus \{0\}$ is the defect coordinate subspace.

Proof. Let $u_k(\alpha) = (x_k(\alpha), y_k(\alpha))^\top$ be the eigenvector for $\mu_k(\alpha)$, normalized so $|x_k(\alpha)| = 1$. The lower block equation gives $y_k = (\mu_k I - Q_\alpha)^{-1} P_\alpha x_k$, and since $|\mu_k| \sim \alpha^{-1}$,

$$|y_k| = O(\alpha).$$

The upper Schur equation, multiplied by α, is $(\nu_k(\alpha) I_m + \Gamma + \alpha E_\alpha(\nu_k(\alpha)))x_k = 0$. Since $\nu_k(\alpha) \to -\gamma_k$ and γ_k is simple, $x_k(\alpha) \to g_k$, an eigenvector of Γ for γ_k. The vectors $g_1, \ldots, g_m$ form a basis of $\mathbb{C}^m$ since Γ has simple eigenvalues.

Let $X_\alpha = (x_1(\alpha), \ldots, x_m(\alpha))$ and $Y_\alpha = (y_1(\alpha), \ldots, y_m(\alpha))$. Then $X_\alpha \to G = (g_1, \ldots, g_m)$, which is invertible, so $|X_\alpha^{-1}| = O(1)$ for small α. The unstable subspace is the graph $\mathcal{U}_\alpha = \{(x, K_\alpha x) : x \in \mathbb{C}^m\}$ with $K_\alpha = Y_\alpha X_\alpha^{-1}$, and $|K_\alpha| = O(\alpha)$.

The projector onto the graph of K_α is

$$P_{\mathcal{U}_\alpha} = \begin{pmatrix} (I + K_\alpha^* K_\alpha)^{-1} & (I + K_\alpha^* K_\alpha)^{-1} K_\alpha^* \\ K_\alpha (I + K_\alpha^* K_\alpha)^{-1} & K_\alpha (I + K_\alpha^* K_\alpha)^{-1} K_\alpha^* \end{pmatrix}.$$

Since $|K_\alpha| = O(\alpha)$, $(I + K_\alpha^* K_\alpha)^{-1} = I + O(\alpha^2)$, so

$$P_{\mathcal{U}_\alpha} - P_{\mathcal{C}} = \begin{pmatrix} O(\alpha^2) & O(\alpha) \\ O(\alpha) & O(\alpha^2) \end{pmatrix},$$

giving $|P_{\mathcal{U}_\alpha} - P_{\mathcal{C}}| = O(\alpha)$. □

Remark 6.3. *Theorem 6.1 reduces to Theorem 2.2 when* $m = 1$, $\Gamma = \lambda + O(\alpha)$ *(a scalar, the leading-order defect coefficient from the cut-cell diagonal), and the block structure matches the one-cut-cell upwind operator. The one-cut-cell results are the base case.*

7 Verification for Raw Cut-Cell Finite-Volume Operators

A general explicit finite-volume update on a cut cell c of volume $|C_c| = \alpha h^d$ takes the form

$$U_c^{n+1} = U_c^n - \frac{\Delta t}{\alpha h^d} \sum_{f \subset \partial C_c} |f| \, F_f. \tag{26}$$

All coefficients in the cut-cell row scale as $O(\alpha^{-1})$; regular cell rows have coefficients $O(1)$. Whether such an operator admits a global singular-block decomposition (23) depends on the specific scheme and geometry. The following result shows that whenever such a decomposition holds, the NIF condition is automatic and Theorems 6.1 and 6.2 apply.

The main result of this section is that the defect matrix Γ is computable directly from the linearized flux coefficients at the cut-cell faces, with no global matrix assembly required.

Theorem 7.1 (Defect matrix formula). *Consider a scalar conservation law* $u_t + f(u)_x = 0$ *on a one-dimensional periodic cut-cell mesh with one cut cell* c *of volume fraction* α. *Let the numerical fluxes at the cut-cell faces be linearized as*

$$F_{c+1/2}(U) = \sum_j a_{c+1/2,j} U_j, \qquad F_{c-1/2}(U) = \sum_j a_{c-1/2,j} U_j, \tag{27}$$

with finite stencil and bounded flux coefficients:

$$a_{c\pm 1/2,j} = O(1) \quad \text{uniformly as } \alpha \to 0. \tag{28}$$

Assume regular cell rows have $O(1)$ *coefficients. Then the linearized cut-cell update operator satisfies Assumption* **??** *with defect matrix*

$$\Gamma = \frac{\Delta t}{h} \big(a_{c+1/2,c} - a_{c-1/2,c} \big), \tag{29}$$

and block scalings $|R_\alpha| = O(1)$, $|B_\alpha| = O(\alpha^{-1})$, $|P_\alpha| = O(1)$, $|Q_\alpha| = O(1)$.

Proof. The conservative cut-cell update is

$$U_c^{n+1} = U_c^n - \frac{\Delta t}{\alpha h} \big(F_{c+1/2}(U) - F_{c-1/2}(U) \big).$$

Substituting (27):

$$U_c^{n+1} = U_c^n - \frac{\Delta t}{\alpha h} \sum_j \big(a_{c+1/2,j} - a_{c-1/2,j} \big) U_j.$$

The coefficient of U_c is $1 - (\Delta t/\alpha h)(a_{c+1/2,c} - a_{c-1/2,c})$. The singular $O(\alpha^{-1})$ part is $-\Gamma/\alpha$ with Γ as in (29); the bounded $+1$ is absorbed into R_α. For $j \neq c$, the cut-cell row coefficients are $(\Delta t/\alpha h)(a_{c+1/2,j} - a_{c-1/2,j}) = O(\alpha^{-1})$ by (28), giving $|B_\alpha| = O(\alpha^{-1})$. Regular rows are divided by h, not αh, so their coefficients are $O(1)$, giving $|P_\alpha| = O(1)$ and $|Q_\alpha| = O(1)$. ☐

Remark 7.2. *The formula* (29) *says* Γ *is the net linearized diagonal flux coefficient at the cut cell, scaled by* $\Delta t/h$*. It is a local two-number computation: evaluate the flux Jacobian at the two faces of the cut cell, extract the diagonal entries* $a_{c+1/2,c}$ *and* $a_{c-1/2,c}$*, and subtract. No global matrix assembly, no eigenvalue computation. The instability threshold* $\Gamma/\alpha > 2$ *and the stability criterion* $s_{\text{stab}} = (\lambda_\alpha - 2)/(\lambda_\alpha - 1) + O(\alpha)$ *with* $\lambda_\alpha = \Gamma/\alpha$ *then follow from Theorems 6.1 and 8.3.*

Remark 7.3. *The three schemes verified in this paper are special cases.* First-order upwind: $a_{c+1/2,c} = a$, $a_{c-1/2,c} = 0$*, giving* $\Gamma = \lambda$*.* Scalar Roe/Godunov (positive speed): $a_{c+1/2,c} = a_{c+1/2}$, $a_{c-1/2,c} = 0$*, giving* $\Gamma = (\Delta t/h)a_{c+1/2}$. MUSCL with bounded stencil: $a_{c+1/2,c} = a \cdot r_c^+$, $a_{c-1/2,c} = a \cdot r_c^-$*, giving* $\Gamma = \lambda(r_c^+ - r_c^-)$*. In all cases* Γ *is read off directly from local flux geometry.*

Remark 7.4. *The bounded-flux condition* (28) *is essential. If a reconstruction uses a slope divided by the cut-cell width* αh*, the flux coefficients grow as* $O(\alpha^{-1})$*, producing* $O(\alpha^{-2})$ *terms in the cut-cell row. The singular-block framework does not apply in that case, and the defect matrix is no longer* $O(1)$*.*

Corollary 7.5 (NIF condition)**.** *Under the conditions of Theorem 7.1, Theorems 6.1 and 6.2 apply, and the NIF condition holds automatically.*

Proof. For large eigenvalues $|\mu| = O(\alpha^{-1})$, $|(\mu I - Q_\alpha)^{-1}| = O(\alpha)$, so $|B_\alpha(\mu I - Q_\alpha)^{-1} P_\alpha| = O(\alpha^{-1}) \cdot O(\alpha) \cdot O(1) = O(1) = o(\alpha^{-1})$. □

The defect matrix Γ is determined by the cut-cell flux geometry and its eigenvalues determine the full unstable spectrum under Theorem 7.1.

8 Geometry-Based Stability Criterion

The singular-block framework yields a practical stability criterion for the blended scheme $A(s) = (1 - s)A + sA_{\text{SRD}}$ that requires no eigenvalue computation on the full mesh.

8.1 The SRD Defect Matrix

Before deriving the criterion, we establish what SRD does to the defect block.

Proposition 8.1 (SRD zeroes the defect block)**.** *For the merge-left SRD operator, the defect block satisfies*

$$\Gamma_{\text{SRD}} = O(\alpha). \tag{30}$$

Full SRD eliminates the leading-order defect to within $O(\alpha)$*.*

Proof. The SRD merged average at cut cell c is $\hat{Q} = (U_{c-1}^{\text{base}} + \alpha U_c^{\text{base}})/(1 + \alpha)$. The coefficient of U_c^n in $\hat{Q}$ — which is the (c, c) entry of A_{SRD} — is

$$(A_{\text{SRD}})_{cc} = \frac{\alpha(1 - \lambda_\alpha)}{1 + \alpha} = \frac{\alpha - \lambda}{1 + \alpha}.$$

In the limit $\alpha \to 0$ with $\lambda_\alpha = \lambda/\alpha$ fixed, $\lambda = \alpha\lambda_\alpha \to 0$, so $(A_{\text{SRD}})_{cc} = (\alpha - \alpha\lambda_\alpha)/(1 + \alpha) = O(\alpha)$. The defect block is the $O(\alpha^{-1})$ part of this entry. Since $(A_{\text{SRD}})_{cc} = O(\alpha)$ in this limit, there is no $O(\alpha^{-1})$ term and $\Gamma_{\text{SRD}} = O(\alpha)$. □

Remark 8.2. *This is the spectral reason SRD stabilizes. The base scheme has defect block* $\Gamma = \lambda + O(\alpha) = O(1)$*, producing an unstable eigenvalue* $\mu_u = 1 - \Gamma/\alpha + O(1) = 1 - \lambda_\alpha + O(1)$*. Full SRD reduces* $\Gamma_{\text{SRD}} = O(\alpha)$*, so its unstable eigenvalue is only* $O(1)$ *rather than* $O(\alpha^{-1})$*. The blended defect block* $\Gamma(s) = (1 - s)\Gamma + s\Gamma_{\text{SRD}} = (1 - s)\lambda + O(\alpha)$ *interpolates between the two, giving* $\mu_u(s) = -(1 - s)\lambda_\alpha + O(1)$*.*

8.2 Stability Criterion

Theorem 8.3 (Geometry-based stability criterion)**.** *As $\alpha \to 0$ with $\lambda_\alpha = \lambda/\alpha$ fixed, the unstable eigenvalue of the blended scheme satisfies*

$$\mu_u(s) = -(\lambda_\alpha - 1)(1 - s) + O(\alpha). \tag{31}$$

The blended scheme is spectrally stable to leading order if and only if

$$s \geq s_{\text{stab}}(\lambda_\alpha) = \frac{\lambda_\alpha - 2}{\lambda_\alpha - 1} + O(\alpha). \tag{32}$$

In particular, $s_{\text{stab}} = 0$ when $\lambda_\alpha = 2$ and $s_{\text{stab}} \to 1$ as $\lambda_\alpha \to \infty$.

Proof. Consider the $(c-1, c)$ two-cell block of $A(s)$, taking $\alpha \to 0$ with $L = \lambda_\alpha$ fixed (so $\lambda = \alpha L \to 0$). The base block on (U_{c-1}, U_c) is

$$M_0 = \begin{pmatrix} 1 - \alpha L & 0 \\ L & 1 - L \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ L & 1 - L \end{pmatrix}.$$

The SRD merged average $\hat{Q} = (U_{c-1}^{\text{base}} + \alpha U_c^{\text{base}})/(1 + \alpha)$ has coefficients on (U_{c-1}, U_c) equal to $(1/(1 + \alpha), \alpha(1 - L)/(1 + \alpha))$, giving

$$M_{\text{SRD}} = \begin{pmatrix} \frac{1}{1+\alpha} & \frac{\alpha(1-L)}{1+\alpha} \\ \frac{1}{1+\alpha} & \frac{\alpha(1-L)}{1+\alpha} \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

The blended block $M_s = (1 - s)M_0 + sM_{\text{SRD}}$ satisfies, as $\alpha \to 0$,

$$M_s \to \begin{pmatrix} 1 & 0 \\ (1 - s)L + s & (1 - s)(1 - L) \end{pmatrix}.$$

The eigenvalues of this lower-triangular matrix are $\mu_1 = 1$ (conservation mode) and $\mu_u(s) = (1 - s)(1 - L) = -(1 - s)(\lambda_\alpha - 1)$, giving (31). Stability $|\mu_u(s)| \leq 1$ then gives (32). □

Remark 8.4. *The formula* (32) *is an asymptotic threshold valid as $\alpha \to 0$ with λ_α fixed. For finite α, the exact stability threshold requires solving the 2×2 characteristic polynomial of $M_s(\alpha)$, which depends on λ, α, and s individually rather than through λ_α alone. Numerically the $O(\alpha)$ error is below* 0.01 *for $\alpha \leq 0.1$, so the formula is accurate at practical cut-cell fractions.*

Remark 8.5. *The formula is computable from geometry alone: given α and λ, compute $\lambda_\alpha = \lambda/\alpha$ and evaluate s_{stab}. No solution data, no eigenvalue computation on the full mesh. For a mesh with m cut cells, the cost is $O(m)$ at mesh generation time. The formula is monotone in λ_α: $s_{\text{stab}} = 0$ at $\lambda_\alpha = 2$ (no redistribution needed) and $s_{\text{stab}} \to 1$ as $\lambda_\alpha \to \infty$ (full redistribution required).*

9 Conclusion

Cut-cell instability is a low-dimensional geometric phenomenon. One small cut cell produces one localized unstable mode. The mode localizes at the cut cell with $O(\alpha)$ decay in distance. State redistribution acts asymptotically in the direction of that mode, with cosine alignment $1 - O(\alpha)$. The stabilization mechanism is spectral: SRD targets the defect subspace directly.

The singular-block framework, for operators admitting the singular-block representation of Assumption **??**, establishes this: the defect matrix Γ, determined by cut-cell geometry, governs the unstable spectrum and the unstable subspace converges to the defect coordinate subspace.

The geometry-based stability criterion (Theorem 8.3) is the practical result: the minimum stabilizing blending parameter is $s_{\text{stab}} = (\lambda_\alpha - 2)/(\lambda_\alpha - 1) + O(\alpha)$, computable from geometry at $O(m)$ cost for m cut cells. This replaces full redistribution at every cut cell with the minimum redistribution needed for stability, reducing unnecessary diffusion near embedded boundaries.

The chain (1) is the conceptual contribution: small-cell geometry determines Γ, which determines $\mathcal{U}$, which determines where stabilization must act. A stabilization operator that aligns with $\mathcal{U}$ requires only one scalar degree of freedom per cut cell to achieve targeted spectral correction. State redistribution satisfies this condition asymptotically. The framework identifies the alignment condition as the criterion for any locally-acting stabilization operator to damp cut-cell instability with minimal intervention.

References

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